Contests with small noise and the robustness of the all-pay auction ☆

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A B S T R A C T

This paper considers all-pay contests in which the relationship between bids and allocations reflects a small amount of noise. Prior work had focused on one particular equilibrium. However, there may be other equilibria. To address this issue, we introduce a new and intuitive measure for the proximity to the all-pay auction. This allows, in particular, to provide simple conditions under which actually *any* equilibrium of the contest is both payoff equivalent and revenue equivalent to the unique equilibrium of the corresponding all-pay auction. The results are shown to have powerful implications for monopoly licensing, political lobbying, electoral competition, optimally biased contests, the empirical analysis of rent-seeking, and dynamic contests.

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1. Introduction

Many types of strategic interaction entail an element of rivalry in which costly resources are irrevocably spent in an attempt to obtain a privileged position compared to others.[1] Hillman and Riley (1989) distinguished two main classes of game-theoretic models of such rivalry. In a *perfectly discriminating* (or deterministic) contest, i.e., in a first-price all-pay auction, the prize is awarded always to the highest bidder.[2] In an *imperfectly discriminating* (or probabilistic) contest, however,

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[1] For example, in a patent race (Loury, 1979; Dasgupta and Stiglitz, 1980), sunk and irreversible research investments determine which firm is more likely to be the first in the market. For an introduction to the literature on contests, see Konrad (2009).

[2] Early work includes Nalebuff and Stiglitz (1983), Moulin (1986), Hillman and Samet (1987), and Hillman and Riley (1989). Baye et al. (1996) accomplished the game-theoretic analysis. The model of the all-pay auction has been applied in various areas such as sales promotion (Varian, 1980; Narasimhan, 1988), competitive screening (Rosenthal and Weiss, 1984), interbank competition (Broecker, 1990), monopoly (Ellingsen, 1991), political lobbying (Baye et al., 1993), and market microstructure (Dennert, 1993). More recent extensions of the theoretical framework include Che and Gale (1998), Clark and Riis (1998a), Konrad (2002), Siegel (2009, 2010), Klose and Kovenock (2015), and Xiao (2016), among others.

a prize is allocated according to a stochastic success function, where a higher bid raises the player's probability of winning—but typically not too strongly.[3] These two classes of models have traditionally been analyzed separately and sometimes with different conclusions.[4] In fact, due to a lack of equilibrium characterizations, most work in the area has been forced to restrict attention to technologies that are either perfectly discriminating or *very* probabilistic, sometimes leaving doubts regarding the robustness of the results.[5]

The first paper that ventured into the no man's land between the two extremes was Baye et al. (1994). They showed that the symmetric two-player Tullock contest with finite $R > 2$ allows a mixed-strategy Nash equilibrium. That equilibrium was found as the limit of equilibria corresponding to a sequence of contests with finite strategy spaces. Since the size of the smallest positive bid in the finite contest turns out to be an upper bound for any player's equilibrium payoff, the limit construction guarantees that rent dissipation in the symmetric equilibrium of the continuous contest must be complete. Che and Gale (2000) addressed related issues in a somewhat different framework. Considering a contest of the difference form with uniform noise, they identified two main classes of equilibria, and proved convergence to the equilibrium of the all-pay auction. For success functions in the tradition of the Tullock contest, however, Che and Gale (2000, p. 23) noted that it was not known if the qualitative properties of the mixed equilibrium in the all-pay auction, such as preemption under heterogeneous valuations, would be similarly preserved if the success function is slightly perturbed.

Substantial progress in this regard has been made by Alcalde and Dahm (2010). Specifically, they identified conditions under which a given probabilistic contest allows an *all-pay auction equilibrium* that, as the term indicates, shares important characteristics with an equilibrium of the corresponding all-pay auction.[6] The construction starts from a symmetric equilibrium with complete rent dissipation in a two-player contest with homogeneous valuations, and subsequently exploits the fact that, in the considered class of contests, a positive bid of any size wins against a zero bid with probability one. Therefore, introducing a mass point at the zero bid of one player (leaving the conditional distribution over positive bids unchanged) implies a proportional reduction in the other player's marginal willingness to pay to win the prize. Since additional, lower-valuation players have little incentive to enter the battle, this indeed allows to construct an all-pay auction equilibrium in the probabilistic contest. As Alcalde and Dahm (2010, p. 5) concluded, however, their results are partial without an improved understanding of the entire equilibrium set. As a matter of fact, it will be shown below that additional, payoff-nonequivalent equilibria exist if the probabilistic contest is not sufficiently "close" to the all-pay auction.

In this paper, the proximity of a given probabilistic contest to the all-pay auction is measured by the extent to which a player's odds of winning, i.e., the ratio between his respective probabilities of winning and losing, is raised by a small percentage change to his own bid. We call this measure *decisiveness*, and show that it is closely related to a variety of existing concepts. Our central result says that, if a probabilistic contest is sufficiently decisive, and if the corresponding all-pay auction has a unique equilibrium, then actually *any* equilibrium of the probabilistic contest is an all-pay auction equilibrium and is, consequently, both payoff-equivalent and revenue-equivalent to the corresponding all-pay auction. Additional results cover the remaining non-generic cases in which the corresponding all-pay auction allows either multiple revenue-equivalent equilibria or multiple revenue-inequivalent equilibria. Finally, regarding the robustness of the all-pay auction, it is shown that payoffs resulting from any equilibrium in the probabilistic contest converge to the unique payoff profile of the corresponding all-pay auction as the decisiveness of the contest exceeds all bounds. Taken together, these results offer a quite comprehensive characterization of the equilibrium set, and thereby resolve the above-mentioned issues. Moreover, powerful conclusions can be drawn in a wide range of specific applications.

The analysis proceeds as follows. After introducing a class of probabilistic n-player contests with potentially heterogeneous valuations, it is shown first that a mixed-strategy Nash equilibrium exists in any such contest. Then, the equilibrium set is examined with a focus on the minimum of the support of the players' bid distributions, as previously done by Baye et al. (1994) and Alcalde and Dahm (2010) in discrete settings.[7] In that part of the analysis, to accomplish the step from homogeneous to heterogeneous valuations, we essentially reverse the construction of Alcalde and Dahm (2010). However, we also work with a couple of new inequalities for winning probabilities when bids are close to each other. Finally, the tools developed mainly for the characterization of the equilibrium set are re-used to establish the above-mentioned robustness property of the all-pay auction.

The equilibrium characterization obtained for any sufficiently decisive n-player contest provides additional and strong support for the view that the deterministic relationship between bids and allocations assumed in the all-pay auction is a useful simplification, in the sense that a probabilistic element could be introduced into that relationship without affecting the equilibrium prediction too much. In a similar vein, the all-pay auction is known to be robust with respect to the introduction of private information. Indeed, as Amann and Leininger (1996) have shown, a symmetric two-player all-pay

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[3] See, in particular, Rosen (1986), Nti (1997), and Cornes and Hartley (2005).

[4] For example, while in an all-pay auction, it may be optimal for a revenue-maximizing politician to exclude the lobbyist with the highest valuation (Baye et al., 1993), this is never the case for the lottery contest (Fang, 2002).

[5] Cf. Konrad and Kovenock (2009, p. 259): "A full characterization of equilibrium for contests with small noise remains, up to now, an open question."

[6] If the corresponding all-pay auction has a unique equilibrium, then an all-pay auction equilibrium is equivalent to that equilibrium in terms of participation probabilities, average bid levels, winning probabilities, expected payoffs, and expected revenue. Otherwise, the definition requires equivalence to an equilibrium in which all but two of the strongest players remain passive.

[7] For a discussion of the related analysis of Klumpp and Polborn (2006), see the applications section.

auction with independent types has a unique Bayesian equilibrium that converges to the complete information outcome as the distributions of valuations degenerate.

The remainder of this paper is organized as follows. Section 2 introduces the set-up. Existence of a mixed-strategy equilibrium is established in Section 3. Section 4 contains the equilibrium characterization. Section 5 deals with the robustness of the all-pay auction. Multiple equilibria are discussed in Section 6. Section 7 offers applications. Section 8 concludes. All proofs have been collected in an Appendix.

2. Set-up and notation

We consider a set-up that is familiar from the standard analysis of the first-price all-pay auction (Baye et al., 1996), yet with the modification that the highest bid need not win with certainty.[8] Thus, there are $n \geqslant 2$ players $i = 1, ..., n$ that simultaneously and independently choose a bid $b_i \geqslant 0$. Player i's valuation of the prize, denoted by V_i, is assumed to be positive. Without loss of generality, players may then be renamed such that $V_1 \geqslant V_2 \geqslant ... \geqslant V_n > 0$. By a *contest success function* Ψ, or CSF in short, we mean a vector of functions $\Psi_i : \mathbb{R}_+^n \to [0, 1]$, one for each player $i = 1, ..., n$, such that

$$\sum_{i=1}^{n} \Psi_i(b_1, ..., b_n) = 1 \tag{1}$$

for any profile of bids $b = (b_1, ..., b_n) \in \mathbb{R}_+^n$.[9] Player i's payoff is given by[10]

$$u_i(b_i, b_{-i}) = \Psi_i(b_i, b_{-i}) V_i - b_i. \tag{2}$$

It is easy to see that bids exceeding V_i are strictly dominated by a zero bid. Thus, it may be assumed without loss of generality that each player $i = 1, ..., n$ chooses his bid from the compact interval $B_i = [0, V_i]$. Given valuations $(V_1, ..., V_n)$, we will refer to the resulting non-cooperative game as the *n-player contest* with CSF Ψ.

The following four assumptions will be imposed on the contest technology.

Assumption 1 *(Monotonicity).* $\Psi_i(b_i, b_{-i})$ is weakly increasing in b_i, for any $i \in \{1, ..., n\}$ and any $b_{-i} \in \mathbb{R}_+^{n-1}$; moreover, $\Psi_{i,j}(b_i, b_j, b_{-i,j})$ is weakly declining in b_j, for any $i, j \in \{1, ..., n\}$ such that $i \neq j$, any $b_i \in \mathbb{R}_+$, and any $b_{-i,j} \in \mathbb{R}_+^{n-2}$.

Assumption 2 *(Zero bids).* $\Psi_i(0, b_{-i}) = 0$ for any $i \in \{1, ..., n\}$ and any $b_{-i} \neq \mathbf{0}_{-i}$; moreover, $\Psi_i(b_i, b_{-i}) > 0$ for any $i \in \{1, ..., n\}$, any $b_i > 0$, and any $b_{-i} \in \mathbb{R}_+^{n-1}$.

Assumption 3 *(Anonymity).* $\Psi_i(b) = \Psi_{\varphi(i)}(b_{\varphi(1)}, ..., b_{\varphi(n)})$ for any $i \in \{1, ..., n\}$, any $b \in \mathbb{R}_+^n$, and any permutation $\varphi : \{1, ..., n\} \to \{1, ..., n\}$.

Assumption 4 *(Smoothness).* Ψ_i is continuous on $\mathbb{R}_+^n \setminus \{\mathbf{0}\}$, for any $i \in \{1, ..., n\}$; moreover, the partial derivative $\partial \Psi_i(b_i, b_{-i}) / \partial b_i$ exists and is continuous in b_i, for any $i \in \{1, ..., n\}$, any $b_i > 0$, and any $b_{-i} \in \mathbb{R}_+^{n-1}$.

Assumption 1 requires a player's probability of winning to be monotone increasing in his own bid, and monotone decreasing in the bid of any other player. Assumption 2 says that a zero bid never wins against a positive bid, whereas a positive bid always has a positive probability of winning. This assumption indeed imposes a certain structure on the CSF.[11] Assumption 3 says that any player's probability to win does not depend on his identity, i.e., not on his index $i \in \{1, ..., n\}$, but only on his bid and on the unordered vector of bids of the other players. This assumption can be relaxed to a certain extent. In particular, as will be explained, our results extend directly to the biased Tullock contest (which is not anonymous).[12] Finally, smoothness is assumed for convenience, and could probably be relaxed. For example, the conclusions of Propositions 1 through 5 below hold for the all-pay auction even though its technology does not satisfy Assumption 4.

The set-up is illustrated by the following examples that will be taken up later as well.

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[8] An even more flexible set-up is considered in the working paper version (Ewerhart, 2014). The present article corresponds roughly to the second part of that paper. Using the standard set-up of a probabilistic contest right from the beginning simplifies the exposition and allows us to do justice to the specific implications of our arguments in the context of the all-pay auction.

[9] The axiomatic approach to the CSF has been pioneered by Skaperdas (1996). For an overview of recent developments, see Jia et al. (2013).

[10] Here and in the subsequent development, it will be convenient to use the notation $\Psi_i(b) = \Psi_i(b_i, b_{-i}) = \Psi_{i,j}(b_i, b_j, b_{-i,j})$, where $b_{-i} = (b_1, ..., b_{i-1}, b_{i+1}, ..., b_n)$ is the profile of bids of all players except i, and $b_{-i,j} = b_{-j,i} = (b_1, ..., b_{i-1}, b_{i+1}, ..., b_{j-1}, b_{j+1}, ..., b_n)$ is the profile of bids of all players except i and j, with $j \neq i$. For better readability, we will also use the symbols $\mathbf{0}$, $\mathbf{0}_{-i}$ and $\mathbf{0}_{-i,j}$ to denote the respective vectors of zero bids in $\mathbb{R}_+^n$, $\mathbb{R}_+^{n-1}$, and $\mathbb{R}_+^{n-2}$.

[11] Specifically, Assumption 2 excludes continuous games that possess slightly different properties. For example, as pointed out by Che and Gale (2000), two players involved in a difference-form contest may each have a small positive equilibrium payoff even when there is very little noise.

[12] In general, however, when the contest is not anonymous, even though one can still find conditions that guarantee that at most one player has a positive equilibrium payoff, it can be more difficult to identify that player, or to pin down the size of the equilibrium rent.

Example 1. Following Tullock (1980), let

$$\Psi_i^{\mathrm{TUL}}(b) \equiv \Psi_i^{\mathrm{TUL}}(b_1, ..., b_n; R) = \frac{b_i^R}{\sum_{j=1}^n b_j^R} \tag{3}$$

if $b = (b_1, ..., b_n) \neq \mathbf{0}$, and $\Psi_i^{\mathrm{TUL}}(\mathbf{0}) = \frac{1}{n}$, where $R > 0$ is the usual parameter.

Example 2. The *ratio-form* CSF (Rosen, 1986) is given by

$$\Psi_i^{\mathrm{RAT}}(b) \equiv \Psi_i^{\mathrm{RAT}}(b_1, ..., b_n; h) = \frac{h(b_i)}{h(b_1) + ... + h(b_n)} \tag{4}$$

if $b \neq \mathbf{0}$, and by $\Psi_i^{\mathrm{RAT}}(\mathbf{0}) = \frac{1}{n}$ otherwise, where $h : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing, differentiable, and satisfies $h(0) = 0$.

Example 3. The *serial* CSF (Alcalde and Dahm, 2007) is given by

$$\Psi_i^{\mathrm{SER}}(b) \equiv \Psi_i^{\mathrm{SER}}(b_1, b_2; \alpha) = \begin{cases} \frac{1}{2}(b_i/b_j)^\alpha & \text{if } b_i < b_j \\[2mm] 1 - \frac{1}{2}(b_j/b_i)^\alpha & \text{if } b_i \geqslant b_j, \end{cases} \tag{5}$$

if $b \neq (0, 0)$, and by $\Psi_i^{\mathrm{SER}}(0, 0) = \frac{1}{2}$ otherwise, where $\alpha > 0$ is a scale parameter.[13]

Example 4. In a simultaneous electoral competition (Snyder, 1989; Klumpp and Polborn, 2006), candidates i's probability of winning a majority of an odd number of $(2s + 1)$ districts may be specified as

$$\Psi_i^{\mathrm{MAJ}}(b) \equiv \Psi_i^{\mathrm{MAJ}}(b_1, b_2; s, R) \tag{6}$$

$$= \sum_{t=s+1}^{2s+1} \binom{2s+1}{t} (\Psi_i^{\mathrm{TUL}}(b, R))^t (1 - \Psi_i^{\mathrm{TUL}}(b, R))^{2s+1-t}, \tag{7}$$

where the binomial coefficient in (7) is defined as usual by $\binom{2s+1}{t} = \frac{(2s+1)!}{t!(2s+1-t)!}$. This technology will be referred to as the *majority* CSF.[14]

3. Existence

This section introduces some terminology regarding randomized strategies and establishes existence of a mixed-strategy Nash equilibrium for the considered class of contests.

Following Dasgupta and Maskin (1986), a *mixed strategy* for player i is a probability measure μ_i on the interval $B_i = [0, V_i]$. Thus, for any measurable set $Y_i \subseteq [0, V_i]$, the real number $\mu_i(Y_i)$ is the probability that the bid realization chosen by player i is contained in Y_i. We write $D(B_i)$ for the set of player i's mixed strategies. As usual, pure strategies may be considered as degenerate probability measures. The support of a mixed strategy $\mu_i \in D(B_i)$ will be denoted by $S(\mu_i)$. Since the support is contained in the compact interval $B_i = [0, V_i]$, we may define $\underline{b}_i = \min S(\mu_i)$ and $\bar{b}_i = \max S(\mu_i)$. Mass points at the zero bid level will play an important role in the analysis. We refer to $\pi_i = 1 - \mu_i(\{0\})$ as player i's *probability of participation*. Player i will be called *passive* if $\pi_i = 0$, *active* if $\pi_i > 0$, and *always active* if $\pi_i = 1$. If some player i is active, then his *lowest positive bid* will be defined as $\underline{b}_i^+ = \inf(S(\mu_i) \backslash \{0\}) \geqslant 0$. If $\underline{b}_i^+ = 0$, then player i will be said to use *arbitrarily small positive bids*.

A *mixed equilibrium* is an n-tuple $\mu^* = (\mu_1^*, ..., \mu_n^*)$, with $\mu_i^* \in D(B_i)$, such that each player maximizes his ex-ante expected payoff, i.e., such that

$$E[u_i(b_i, b_{-i}) | \mu_i^*, \mu_{-i}^*] = \max_{\mu_i \in D(B_i)} E[u_i(b_i, b_{-i}) | \mu_i, \mu_{-i}^*], \tag{8}$$

for any $i = 1, ..., n$, where $\mu_{-i}^* = (\mu_1^*, ..., \mu_{i-1}^*, \mu_{i+1}^*, ..., \mu_n^*)$. For a given equilibrium μ^*, denote by $p_i^* = E[\Psi_i(b_i, b_{-i}) | \mu_i^*, \mu_{-i}^*]$ and $b_i^* = E[b_i | \mu_i^*]$, respectively, player i's ex-ante probability of winning and player i's average bid level. Player i's equilibrium payoff, or *rent*, is then given as $u_i^* = p_i^* V_i - b_i^*$. Clearly, because of the option to bid zero, $u_i^* \geqslant 0$. Moreover,

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[13] For expositional reasons, we restrict attention to the case of two players. Lemma 2 below holds, however, also for the serial contest with any finite number of players.

[14] The assumption that bidders allocate resources equally across districts will be relaxed later in the paper. Similarly, we will drop the assumption that the district CSF is of the Tullock form.

$u_i^* = E[u_i(b_i, b_{-i}) | \mu_{-i}^*]$ for any $b_i \in S(\mu_i^*)\backslash\{0\}$.[15] Finally, the *expected revenue* is $\mathcal{R} = \sum_{i=1}^n b_i^*$, i.e., the total of players' average bid levels.

To prove existence of a mixed equilibrium, one notes that any contest with heterogeneous valuations is strategically equivalent to a contest with homogeneous valuations but heterogeneous marginal costs. The latter case, however, has been dealt with in prior work by Bagh (2010).[16]

Lemma 1. *A mixed equilibrium μ^* exists in any n-player contest.*

4. Equilibrium characterization

This section contains the central parts of the analysis. We start by introducing, on the set of CSFs, a measure of proximity to the technology of the all-pay auction. Then, after outlining the main argument, the equilibrium characterizations are presented, first for two players and subsequently for more than two players.

4.1. Measuring the proximity to the all-pay auction

For a given CSF Ψ, the ratio $Q_i = \Psi_i/(1 - \Psi_i)$ will be referred to as player i's *odds of winning*. At a bid vector $(b_i, b_{-i}) \in \mathbb{R}_+^n$, the own-bid elasticity of player i's odds of winning is given as

$$\rho_i = \frac{\partial \ln Q_i}{\partial \ln b_i} = \frac{b_i}{\Psi_i(1 - \Psi_i)} \cdot \frac{\partial \Psi_i}{\partial b_i}. \tag{9}$$

One can convince oneself that, under the present assumptions, $\rho_i(b_i, b_{-i})$ is well-defined if $b_i > 0$ and $b_{-i} \in \mathbb{R}_+^{n-1}\backslash\{\mathbf{0}_{-i}\}$. Define now the *decisiveness* of the CSF Ψ by $\rho \equiv \rho(\Psi) = \inf\{\rho_i(b_i, b_{-i}) : i \in \{1, ..., n\}, b_i > 0, b_{-i} \neq \mathbf{0}_{-i}\}$, i.e., as the joint infimum of the functions ρ_i over the relevant domains.[17]

The following result provides information about the decisiveness of the Tullock, ratio-form, serial, and majority CSFs.

Lemma 2. $\rho(\Psi^{\mathrm{TUL}}) = R$, $\rho(\Psi^{\mathrm{RAT}}) = \inf_{x>0} xh'(x)/h(x)$, $\rho(\Psi^{\mathrm{SER}}) = \alpha$, and $\rho(\Psi^{\mathrm{MAJ}}) = \frac{(2s+1)!}{(s!)^2 4^s} R$.

The lemma illustrates that the decisiveness notion is closely related to a variety of existing concepts. In Tullock's model, the parameter R is an immediate measure for the proximity to the all-pay auction simply because, as $R \to \infty$, the CSF converges pointwise to the technology of the all-pay auction.[18] The fact that the own-bid elasticity of the odds of winning is constant and equal to R is the result of a straightforward computation.[19] More generally, the decisiveness of a contest of the ratio form can be seen to correspond to the infimum of Rosen's (1986) elasticity $\eta(x) = xh'(x)/h(x)$ over all positive bid levels $x > 0$. In the case of the serial contest, the own-bid elasticity of the odds of winning assumes values in the interval $(\alpha, 2\alpha]$, where lower values correspond to more unbalanced bidding. Here, taking the limit in the definition of decisiveness is indeed necessary. Finally, the decisiveness of the majority contest turns out to correspond to the probability of a given voter being pivotal in a random voting model (Penrose, 1946), multiplied with the decisiveness of the individual contest. If the district CSF Ψ is not of the Tullock type, then this relationship generalizes into a useful lower bound, viz. $\rho(\Psi^{\mathrm{MAJ}}) \geqslant \frac{2s+1}{4^s}\binom{2s}{s} \cdot \rho(\Psi) > \sqrt{s} \cdot \rho(\Psi)$.[20] In particular, for any given district CSF Ψ with $\rho(\Psi) > 0$, the majority contest becomes arbitrarily decisive as the number of districts grows large.

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[15] For a proof, see Lemma A.1 in the Appendix. To understand why the zero bid is excluded, think of a two-player all-pay auction with valuations $V_1 > V_2$. For player 1, any positive bid $b_1 \in (0, V_2)$ yields the equilibrium payoff $u_1^* = V_1 - V_2 > 0$, yet the zero bid, even though it is contained in the support $S(\mu_1^*) = [0, V_2]$, yields strictly less, viz. $(V_1 - V_2)/2$. The situation here is similar because the CSF is discontinuous at the origin.

[16] Similar results that do not cover the present situation can be found in Baye et al. (1994), Yang (1994), and Alcalde and Dahm (2010).

[17] As an elasticity, the decisiveness parameter allows the usual graphical interpretation. For instance, in a pure lottery (i.e., a Tullock contest with parameter $R = 0$), the odds of winning are constant and equal to $Q_i = 1/(n-1)$, hence perfectly inelastic. At the other extreme, the all-pay auction, the odds of winning jump from zero to infinity at the highest competing bid, and hence, are perfectly elastic. We are interested here in the case where that elasticity is large but still finite.

[18] Consistent with this property, the parameter R has been interpreted alternatively as a measure of military mass-effect (Hirshleifer, 1989), political culture (Che and Gale, 1997), noise (Jia, 2008), and incomplete information (Eccles and Wegner, 2014), for instance.

[19] Indeed, as noted by Wang (2010, fn. 4), for any $b_i > 0$ and any $b_{-i} \neq \mathbf{0}_{-i}$,

$$\rho_i(b_i, b_{-i}) = \frac{b_i}{\Psi_i^{\mathrm{TUL}}(1 - \Psi_i^{\mathrm{TUL}})} \frac{\partial \Psi_i^{\mathrm{TUL}}}{\partial b_i} = \frac{b_i(\sum_{j=1}^n b_j^R)}{b_i^R(\sum_{j \neq i} b_j^R)} \frac{R b_i^{R-1}(\sum_{j \neq i} b_j^R)}{(\sum_{j=1}^n b_j^R)^2} = R. \tag{10}$$

The claim follows.

[20] For the second inequality, see, e.g., Sasvári (1999).

4.2. Intuitive discussion of the main idea

The main observation of this paper is that a high decisiveness parameter renders competition in an imperfectly discriminating contest nearly as ruthless as in the all-pay auction. To understand why this is so, consider the best-response correspondence in a two-player contest with $\rho > 2$. Suppose that player 2 uses a mixed strategy μ_2^* with a positive lowest bid realization $\underline{b}_2 > 0$. It is claimed that player 1 will then *never* find it optimal to bid in the interval $(0, \underline{b}_2]$. Indeed, suppose that player 1 considers submitting a bid b_1 in that interval. Then, for any b_2 in the support of player 2's bid distribution, $\Psi_1(b_1, b_2) \leqslant \Psi_1(b_2, b_2) = \frac{1}{2}$, by monotonicity and anonymity. Combining this with the fact that the decisiveness exceeds two leads to the key inequality[21]

$$\frac{\partial \Psi_1}{\partial b_1} = \frac{\Psi_1}{b_1} \cdot \underbrace{(1 - \Psi_1)}_{\geqslant 1/2} \cdot \underbrace{\rho_1}_{\geqslant \rho(\Psi) > 2} > \frac{\Psi_1}{b_1}. \tag{12}$$

From here, taking expectations, one arrives at

$$\frac{\partial E\left[u_1 | \mu_2^*\right]}{\partial b_1} = E\left[\frac{\partial \Psi_1}{\partial b_1} \middle| \mu_2^*\right] V_1 - 1 > E\left[\frac{\Psi_1}{b_1} \middle| \mu_2^*\right] V_1 - 1 = \frac{E\left[u_1 | \mu_2^*\right]}{b_1}, \tag{13}$$

which says that, either player 1 incurs an expected loss (viz. if the right-hand side is negative), or player 1 has a strict incentive to raise his bid (viz. if the right-hand side is weakly positive). Thus, precisely as in the all-pay auction, it is never optimal for player 1 to place a bid in the interval $(0, \underline{b}_2]$.

From the above, it follows that a given positive bid $b_1 > 0$ of player 1 can be a best response only if player 2 uses, with some positive probability, bids strictly below b_1. Since this argument remains unchanged when the roles of players 1 and 2 are reversed, there is a sense in which the equilibrium unravels.[22] But, because the CSF is discontinuous at the origin, at least one player must be always active. Thus, the other player cannot have a positive rent. This basic logic generalizes in a straightforward way, and is used in the present paper to show that, in any n-player contest with $\rho > 2$, there is at most one player with a positive equilibrium payoff.

4.3. The case of two players

For the case of two players, the implications of this observation are particularly strong. Specifically, using additional arguments that revert the construction in Alcalde and Dahm (2010), we show that any rent necessarily accrues to player 1 and equals $V_1 - V_2$. As a consequence, it turns out that any equilibrium of the imperfectly discriminating contest shares several key characteristics with the unique equilibrium of the corresponding all-pay auction (Hillman and Riley, 1989, Prop. 2).

Proposition 1. *Consider a two-player contest with CSF Ψ and valuations $V_1 \geqslant V_2 > 0$. Then, provided that $\rho(\Psi) > 2$, any mixed equilibrium μ^* satisfies the following three properties:*

(A1) *Player 1 participates with probability $\pi_1 = 1$, uses arbitrarily small positive bids, bids an average amount of $b_1^* = V_2/2$, wins with probability $p_1^* = 1 - V_2/(2V_1)$, and receives a rent of $V_1 - V_2$.*

(A2) *Player 2 participates with probability $\pi_2 = V_2/V_1$, uses arbitrarily small positive bids, bids an average amount of $b_2^* = (V_2)^2/(2V_1)$, wins with probability $p_2^* = V_2/(2V_1)$, and receives no rent.*

(A3) *The expected revenue from the contest is $\mathcal{R} = \mathcal{R}^{APA} \equiv V_2(V_1 + V_2)/(2V_1)$.*

The conclusions of Proposition 1 indeed match precisely those for the all-pay auction. In particular, rent dissipation is complete in any equilibrium of any two-player contest with homogeneous valuations and $\rho > 2$.[23]

Proposition 1 strengthens the conclusions of Alcalde and Dahm (2010) in two ways. First, properties (A1–A3) are shown to hold for *any* mixed equilibrium of the probabilistic contest, rather than for *some* equilibrium. Second, both players are

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[21] For example, for the Tullock contest with decisiveness $R > 2$, inequality (12) reads

$$\frac{\partial \Psi_1(b_1, b_2)}{\partial b_1} = \frac{b_1^{R-1}}{b_1^R + b_2^R} \cdot \underbrace{\frac{b_2^R}{b_1^R + b_2^R}}_{\geqslant 1/2} \cdot R > \frac{b_1^{R-1}}{b_1^R + b_2^R} = \frac{\Psi_1(b_1, b_2)}{b_1}. \tag{11}$$

Further elaboration on this example as well as many additional results can be found in a companion paper (Ewerhart, 2015). Special thanks goes to Casper de Vries for suggesting the separate documentation of the Tullock case.

[22] I am indebted to Benny Moldovanu for suggesting this analogy.

[23] Uniqueness remains an interesting issue. For example, it would directly imply some results of the present paper. However, proving uniqueness is not straightforward, e.g., because mixed equilibrium strategies are not available in explicit form. Even in cases where equilibrium strategies have been characterized, as in the Tullock case, it is not presently known if the equilibrium is unique for $R > 2$.

shown to use arbitrarily small positive bids. It is easy to see that this excludes, in particular, the use of pure strategies.[24] But also the assumptions of Proposition 1 differ from existing conditions. For example, as shown by Alcalde and Dahm (2010), the two-player Tullock contest with parameter exactly equal to $R = 2$ allows an all-pay auction equilibrium. Yet this case is not covered by Proposition 1. Intuitively, this makes sense because, in the considered equilibrium, at least one of the players uses a pure strategy, which disallows the application of our methods. Similarly, an all-pay auction equilibrium exists in the serial contest with parameter $\alpha = 1$, yet this case is not covered by our arguments.[25] In this sense, the present paper complements the analysis of Alcalde and Dahm (2010).

4.4. More than two players

The characterization of the equilibrium set complicates for $n \geqslant 3$ players. One reason for this is the well-known fact that the all-pay auction allows a continuum of equilibria if $V_2 = V_3$ (Baye et al., 1996). While those equilibria are all payoff-equivalent in the sense that a preemptive rent of $V_1 - V_2$ goes to player 1 whereas all other players earn zero, they need not be equivalent in terms of revenue. The following result starts the analysis of the case of $n \geqslant 3$ players by dealing with precisely those cases in which the corresponding all-pay auction has a unique equilibrium.

Proposition 2. *Consider an n-player contest with CSF Ψ and valuations $V_1 \geqslant V_2 > V_3 \geqslant ... \geqslant V_n > 0$. Then, provided that $\rho(\Psi)$ is sufficiently large, any mixed equilibrium μ^* satisfies properties (A1–A3). Moreover, players $3, ..., n$ remain passive.*

Proposition 2 may be understood as a variation of the corresponding uniqueness result for the all-pay auction (Hillman and Riley, 1989, Prop. 4). The main point to prove is that players $3, ..., n$ remain passive in any equilibrium. This is accomplished by noting that if, say, player 3 was active, then players 1 and 2 could, in a sufficiently decisive contest, each ensure a positive expected payoff by bidding player 3's highest bid realization $\bar{b}_3$. But this would imply that more than one player receives a positive rent, which is impossible, as discussed above. Thus, provided that $V_2 > V_3$, and that ρ is sufficiently large, all but the two strongest players remain passive. However, as will be explained later in the paper, this conclusion need not hold when the decisiveness is merely larger than two.

Suppose, next, that at least three contestants share the highest valuation. In this case, the all-pay auction allows a continuum of equilibria, all of which are still equivalent in terms of payoffs and revenues. Following Baye et al. (1996), we are also interested in the case of "symmetric" equilibria, i.e., equilibria in which players with identical valuations use identical mixed strategies.

Proposition 3. *Assume that $V_1 = ... = V_m > V_{m+1} \geqslant ... \geqslant V_n > 0$, where $m \geqslant 3$. Then, for ρ sufficiently large, in any mixed equilibrium μ^*, there is a player $i \in \{1, ..., m\}$ such that $u_j^* = 0$ for any $j \neq i$, players $m + 1$ through n remain passive, and the expected revenue from the contest satisfies $\mathcal{R} = V_1 - u_i^*$. In particular, all "symmetric" equilibria entail complete rent-dissipation.*

Thus, if more than two players share the same, highest valuation, then all players with a strictly lower valuation remain passive, at most one of the active players earns a positive rent, and that rent reduces the expected revenue in a zero-sum fashion compared to full dissipation. In particular, payoff and revenue equivalence holds under the assumptions of Proposition 3 when attention is restricted to "symmetric" equilibria. For equilibria that are not "symmetric", however, our findings leave the theoretical possibility that one player has a small positive rent even if ρ is large.[26]

Finally, assume that one strong player fights against at least two weaker players that share the same valuation. For the all-pay auction, payoff equivalence continues to hold, yet revenue equivalence breaks down. For the probabilistic contest, one can still show that the strongest player receives a positive rent, while all other players earn zero. Moreover, as in the all-pay auction, the expected revenue from any equilibrium is strictly lower than the second-highest valuation.

Proposition 4. *Assume that $V_1 > V_2 = ... = V_m > V_{m+1} \geqslant ... \geqslant V_n > 0$, where $m \geqslant 3$. Then, for ρ sufficiently large, in any mixed equilibrium μ^*, player 1 participates with probability $\pi_1 = 1$ and receives a positive rent $u_1^* > 0$, all other players have a zero rent $u_2^* = ... = u_n^* = 0$, players $m + 1$ through n remain passive, and the expected revenue from the contest satisfies $\mathcal{R} = V_2 + p_1^*(V_1 - V_2) - u_1^* < V_2$.[27]*

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[24] As noted by Rosen (1986), a pure-strategy equilibrium cannot exist in any two-player contest of the ratio form if the elasticity η exceeds two at the local equilibrium. Similarly, it follows from our results that a pure-strategy equilibrium cannot exist in any n-player contest with $\rho > 2$.

[25] However, the cases $\alpha \in (1, 2]$ can be dealt with. This is because, in Proposition 1, the condition on the decisiveness may be replaced by the somewhat weaker assumption that the own-bid elasticity of the probability of winning for the player with the weakly lower bid is strictly larger than one.

[26] Numerical computations suggest that this possibility might indeed be a real one. E.g., a three-player Tullock contest allows staggering equilibria similar to those identified by Che and Gale (2000). However, if two or more players are always active, then rent dissipation is necessarily complete, and each of the always active players uses arbitrarily small positive bids, just as in the all-pay auction.

[27] For expositional clarity, we mention that the last inequality in the statement of Proposition 4, i.e., $\mathcal{R} < V_2$, relies on the robustness result that will be stated and proved in the next section.

To elucidate the expression for the expected revenue somewhat, recall that in the all-pay auction, $u_1^* = V_1 - V_2$ and $p_1^* = \frac{V_1 - V_2 + b_1^*}{V_1}$. Plugging this into the equation above yields, after some manipulation, the well-known expression for the expected revenue in the n-player all-pay auction, $\mathcal{R}^{\text{APA}} = \frac{V_2}{V_1} V_2 + (1 - \frac{V_2}{V_1}) b_1^*$. Thus, like the previous result, Proposition 4 leaves, compared to the prediction for the all-pay auction, one additional degree of freedom.[28] Overall, while Propositions 3 and 4 are somewhat less explicit than their counterparts for the all-pay auction, powerful conclusions remain feasible, as will be shown in the applications section.

5. Robustness of the all-pay auction

We arrive at the promised robustness result.

Proposition 5. *Fix arbitrary valuations $V_1 \geqslant V_2 \geqslant V_3 \geqslant ... \geqslant V_n > 0$, and $\delta > 0$. Then, provided that ρ is sufficiently large, any mixed equilibrium μ^* satisfies $\left| u_1^* - (V_1 - V_2) \right| < \delta$, and $u_j^* < \delta$ for $j = 2, ..., n$.*

Thus, for any finite number of players and arbitrary valuations, a small change in the technology of the all-pay auction does not affect equilibrium payoffs very much. For example, for the Tullock technology, Proposition 5 assures us that, for any given ordered set of valuations $V_1 \geqslant V_2 \geqslant V_3 \geqslant ... \geqslant V_n > 0$, and for any given level of precision $\delta > 0$, there is a parameter value R^* such that for any $R > R^*$, the expected payoff of any player $i = 1, ..., n$ in any mixed equilibrium of the probabilistic contest is δ-close to i's expected payoff in the corresponding all-pay auction. Payoff equivalence may be reached for some or all players already at a finite ρ, as follows from Propositions 1 through 4. In general, however, the approximation seems necessary.[29]

The proof combines several arguments. First, it will be intuitively clear that, in a sufficiently decisive contest, player 1 can secure a rent arbitrary close to $V_1 - V_2$. After all, player 1's probability of winning from overbidding all his competitors by a small increment approaches one as the contest becomes very decisive. Conversely, in a very decisive contest, player 2 need not accept that player 1 gets away with a rent substantially larger than $V_1 - V_2$, because that would imply that player 1 bids strictly below V_2 *with probability one*, and hence could be overbid. Indeed, it turns out that, even if there are additional bidders, such overbidding yields winning probabilities arbitrarily close to the probability that player 1 achieves in equilibrium with his highest bid realization $\overline{b}_1$. As a consequence, player 2 would be able to obtain a positive expected payoff from overbidding player 1. But then, two players would have a positive equilibrium payoff, which is impossible. Thus, the limit payoff for player 1 is indeed $V_1 - V_2$, and zero for the other players, as in the corresponding all-pay auction.[30]

In prior work, Che and Gale (2000) obtained a related robustness result for contests of the difference-form. Specifically, as the variance of the uniform noise variable goes to zero, equilibrium bid distributions of the probabilistic contest converge uniformly to the unique equilibrium of the corresponding two-player all-pay auction. Since uniform convergence implies convergence in the mean, this proves revenue equivalence and, in particular, full rent dissipation in the case of homogeneous valuations. The present paper complements their contribution by developing an alternative approach that allows deriving structurally similar results for another important class of contest technologies, but covering also many cases in which an explicit characterization of equilibrium strategies cannot be hoped for.

6. Payoff-nonequivalent equilibria

This section documents an important fact that also serves as a motivation for the present paper. Specifically, it will be shown that, provided that the decisiveness measure is *not* high enough, an n-player contest may possess, in addition to the equilibrium identified by Alcalde and Dahm (2010), another equilibrium that does not resemble any equilibrium of the corresponding all-pay auction. We start with a simple example that illustrates this possibility.

Example 5. Consider a three-player Tullock contest with $R = 2$. Assume that valuations satisfy $V_1/V_3 < \sqrt{2}$, and $V_3/V_2 = 1 - \varepsilon$ for $\varepsilon > 0$ small. Then, there is an equilibrium in which player 1 remains passive, player 2 bids $b_2 = V_3/2$ with probability 1, whereas player 3 chooses $b_3 = V_3/2$ with probability $1 - \varepsilon$ and $b_3 = 0$ with probability ε. Indeed, player 1's expected payoff satisfies

$$E[u_1(b_1, b_{-1}) | \mu_{-1}^*]$$
$$= (1 - \varepsilon) \left(\frac{b_1^2 V_1}{b_1^2 + V_3^2/4 + V_3^2/4} - b_1 \right) + \varepsilon \left(\frac{b_1^2 V_1}{b_1^2 + V_3^2/4} - b_1 \right) \tag{14}$$

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[28] It is conjectured that the equilibrium rent for the strongest player may differ from $V_1 - V_2$ even if attention is restricted to "symmetric" equilibria.

[29] Also, answering a question raised by Wolfgang Leininger, given that the equilibrium characterizations in the limit are tighter than those available for an approximating sequence of probabilistic contests, Proposition 5 unfortunately cannot be used as a tool for equilibrium selection.

[30] Is the expected revenue robust with respect to the introduction of noise? Provided that all equilibria of the corresponding all-pay auction are revenue-equivalent to each other, our earlier results imply that, indeed, Proposition 5 can be enriched by the conclusion that, like expected payoffs, also the expected revenue is δ-close to that of the all-pay auction, i.e., $\left| \mathcal{R} - \mathcal{R}^{\text{APA}} \right| < \delta$. However, it looks harder to get substantially beyond the inequality $\mathcal{R} < V_2$ in the case in which a single strongest player fights against at least two equally strong but weaker opponents.

$$= b_1 \left\{ (1 - \varepsilon) \frac{V_1^2 - 2V_3^2 - (V_1 - 2b_1)^2}{4b_1^2 + 2V_3^2} + \varepsilon \frac{V_1^2 - V_3^2 - (V_1 - 2b_1)^2}{4b_1^2 + V_3^2} \right\} \tag{15}$$

$$\leqslant b_1 \left\{ (1 - \varepsilon) \frac{V_1^2 - 2V_3^2}{4b_1^2 + 2V_3^2} + \varepsilon \frac{V_1^2 - V_3^2}{4b_1^2 + V_3^2} \right\}. \tag{16}$$

By assumption, the first ratio in (16) is negative and the second positive, so that

$$E[u_1(b_1, b_{-1}) | \mu_{-1}^*] \leqslant b_1 \left\{ (1 - \varepsilon) \frac{V_1^2 - 2V_3^2}{4V_1^2 + 2V_3^2} + \varepsilon \frac{V_1^2 - V_3^2}{V_3^2} \right\} \tag{17}$$

for any $b_1 \leqslant V_1$.[31] Taking then the ε sufficiently small, the first term in the brackets dominates the second part. Hence, $b_1 = 0$ is optimal. Moreover, the equilibrium property for players 2 and 3 can be verified exactly as in Alcalde and Dahm (2010, Ex. 3.3). However, since player 2 receives a positive rent, this equilibrium is *not* payoff-equivalent to any equilibrium of the corresponding all-pay auction.

The following result shows more generally that, for any number of players and arbitrarily high levels of decisiveness, there are robust specifications of valuation vectors such that player 1 may find it optimal to stay entirely out of the contest— even if his valuation is strictly higher than the valuation of any other player.

Proposition 6. *For any $R \geqslant 2$ and any $n \geqslant 3$, there is a nonempty open set of valuation vectors $(V_1, ..., V_n)$ such that the corresponding n-player Tullock contest admits a mixed equilibrium μ^* that does not satisfy any of the conditions (A1–A3).*

The proposition above shows that existing equilibrium characterizations for contests with small noise are indeed of a partial nature. The result also helps to see why it is essential to assume that the decisiveness is sufficiently high, rather than higher than two, when the number of contestants exceeds two.[32]

7. Applications

This section illustrates the results by applying them in more specific environments.

7.1. Buyer lobbying

Ellingsen (1991) pointed out that buyer lobbying is socially desirable when a given number of sellers strive to obtain a monopoly license through an all-pay auction. As will be explained now, that policy conclusion remains valid in a somewhat noisy environment. Specifically, without buyer lobbying, there are two or more sellers that have an identical valuation of winning the contest, viz. the monopoly profit $T > 0$ (the "Tullock costs"). From Propositions 1 and 3, for ρ large enough, total rent-seeking expenditures are given by $\mathcal{R}^{\text{without}} = T - u_i^{*,\text{without}}$, where i is the seller that potentially manages to end up with a positive rent $u_i^{*,\text{without}} \geqslant 0$. Since monopoly is certain, the social waste in this case is $\mathcal{W}^{\text{without}} = H + \mathcal{R}^{\text{without}} = H + T - u_i^{*,\text{without}}$, where $H > 0$ denotes "Harberger costs." On the other hand, with lobbying, a buyer organization with valuation $H + T$ enters the contest. By Proposition 4, total rent-seeking expenditures are given now by $\mathcal{R}^{\text{with}} = T + p_1^* H - u_1^{*,\text{with}}$, where $p_1^* > 0$ is the probability that the buyers win the contest, and $u_1^{*,\text{with}}$ is their equilibrium payoff. Social waste in this case is $\mathcal{W}^{\text{with}} = (1 - p_1^*)H + \mathcal{R}^{\text{with}} = T + H - u_1^{*,\text{with}}$. Thus, buyer lobbying is socially desirable if $u_1^{*,\text{with}} > u_i^{*,\text{without}}$. But for the all-pay auction, $u_1^{*,\text{with}} = H > 0 = u_i^{*,\text{without}}$. Hence, from Proposition 5, we may conclude that for ρ large enough, buyer lobbying is indeed socially desirable, even if the political contest is not perfectly discriminating, and regardless of the equilibrium.[33] We have shown:

Corollary 1. *If the contest is sufficiently decisive, buyer lobbying is socially desirable.*

7.2. The exclusion principle

Baye et al. (1993) consider the problem of a revenue-maximizing politician that selects, from a given set of lobbyists with known valuations $V_1 \geqslant V_2 \geqslant ... \geqslant V_n > 0$, a subset of finalists that participate in an all-pay auction. Let $k \in \{1, ..., n\}$ be

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[31] I am grateful to a referee for pointing out a typo in an earlier version.

[32] In an interesting recent paper, Amegashie (2012) has shown that, even in a two-player contest, an all-pay auction equilibrium may coexist with a payoff-nonequivalent equilibrium when the assumptions of the present analysis are violated.

[33] This is even true when, as in Baye et al. (1996), only a proportion $0 \leqslant \lambda \leqslant 1$ of lobbying expenditures is assumed to be socially wasteful.

the *largest* index such that $\mathcal{R}_k \equiv \left(1 + \frac{V_{k+1}}{V_k}\right)\frac{V_{k+1}}{2} \geqslant \left(1 + \frac{V_{i+1}}{V_i}\right)\frac{V_{i+1}}{2}$ for all $i = 1, ..., n$. For the all-pay auction, the maximum expected revenue of the politician is $\mathcal{R}_k$, and that revenue can be implemented by excluding all lobbyists $j \in \{1, ..., k-1\}$.[34] This is equally true for any sufficiently decisive contest.

Corollary 2. *In any sufficiently decisive contest, the politician optimally excludes lobbyists $j \in \{1, ..., k-1\}$, and earns an expected revenue of $\mathcal{R}_k$.*

To see this, denote by $1', 2', ...$ the indices of the finalists, in increasing order. If there are only two finalists, or if there are more than two finalists and $V_{2'} > V_{3'}$, then revenue equivalence to the all-pay auction holds in the final by Propositions 1 and 2. But otherwise, we have either $V_{1'} = V_{2'} = V_{3'}$, which cannot yield more revenue than the corresponding all-pay auction by Proposition 3, or $V_{1'} > V_{2'} = V_{3'}$, which is even strictly suboptimal by Proposition 4 because a final between $2'$ and $3'$ yields a higher expected revenue of $V_{2'}$. Clearly, these considerations imply the corollary above.

7.3. Electoral competition

In the simultaneous electoral competition model (Snyder, 1989; Klumpp and Polborn, 2006), two candidates $i \in \{1, 2\}$ participate in $(2s+1)$ probabilistic district contests. There is a single prize, homogeneously valued at $V > 0$, that is allocated to the candidate that wins the majority of districts. Dropping an earlier assumption, each candidate may now choose a *campaign strategy* $\mathbf{b}_i = (b_i^{(1)}, ..., b_i^{(2s+1)}) \in [0, V]^{2s+1}$, i.e., a separate bid for each district. A mixed campaign strategy is, consequently, a probability measure on $[0, V]^{2s+1}$. For the special case of the Tullock district CSF with $R \leqslant 1$, it has been noted that, in any equilibrium of the electoral competition model, mixed campaign strategies are necessarily *uniform*, in the sense that $b_i^{(1)} = ... = b_i^{(2s+1)}$ holds with probability one. This observation is very valuable because it implies that our earlier results may be used to characterize also the equilibrium set of the much more complicated electoral competition model. For instance, Proposition 1 and Lemma 2 now jointly imply that, for $\rho(\Psi^{\mathrm{MAJ}}) = \frac{(2s+1)!}{(s!)^2 4^s} R > 2$, there is full rent dissipation in any equilibrium of the electoral competition model.[35] One can check (details omitted) that similar conclusions hold provided that the district CSF is of the ratio form with h concave.

Corollary 3. *Let Ψ^{RAT} be a district CSF of the ratio form with concave h and $\rho > 0$. Then any mixed equilibrium in the electoral competition model employs uniform strategies and, provided that $s \geqslant 4/\rho^2$, entails full rent dissipation.*

Thus, the present analysis offers rigorous proofs of existing equilibrium characterizations for the two-candidate electoral competition model, and also extends those results to a somewhat more flexible class of district CSFs.

7.4. Biased Tullock contests

For arbitrary weights $a_1 > 0, ..., a_n > 0$ such that $\sum_{i=1}^n a_i = 1$, let the *biased* Tullock technology Ψ^{BIAS} be given by $\Psi_i^{\mathrm{BIAS}}(\widehat{b}) = a_i \widehat{b}_i^R / \sum_{j=1}^n a_j \widehat{b}_j^R$ if $\widehat{b} \neq \mathbf{0}$, and $\Psi_i^{\mathrm{BIAS}}(\mathbf{0}) = a_i$.[36] Simple substitutions $b_i = a_i^{1/R} \widehat{b}_i$, for $i = 1, ..., n$, transform the biased game with payoffs $\widehat{u}_i(\widehat{b}) = \Psi_i^{\mathrm{BIAS}}(\widehat{b}) \widehat{V}_i - \widehat{b}_i$ into an anonymous contest with CSF $\Psi^{\mathrm{TUL}} = \Psi^{\mathrm{TUL}}(b) = \Psi^{\mathrm{BIAS}}(\widehat{b})$, unordered valuations $V_i = a_i^{1/R} \widehat{V}_i$, and payoffs $u_i(b) = a_i^{1/R} \widehat{u}_i(\widehat{b})$. Thereby, our findings have immediate implications also for biased Tullock contests. For example, Franke et al. (2014) proved that the optimally biased all-pay auction extracts a revenue of $(V_1 + V_2)/2$, and conjectured that this expression is an upper bound also for the Tullock contest. Using Proposition 1, it is straightforward to establish that the biased two-player contest with $R > 2$ is, in fact, revenue-equivalent to the biased all-pay auction, which confirms their conjecture in this case.

Corollary 4. *The optimally biased two-player Tullock contest with $R > 2$ yields an expected revenue of $(V_1 + V_2)/2$.*

7.5. Empirical measures of rent-seeking expenditures

The complete dissipation hypothesis is an important instrument for measuring the welfare losses resulting from rent-seeking activities (Krueger, 1974; Posner, 1975). The following result lends additional support to that hypothesis.

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[34] Baye et al. (1993) exclude all lobbyists with valuations strictly exceeding V_k. Our solution yields the same revenue, but leaves at most two lobbyists with valuation V_k in the final. For example, if $V_1 = 50$, $V_2 = V_3 = V_4 = 40$, and $V_5 = 38$, then our solution excludes lobbyist 1 and 2, rather than only lobbyist 1. Either way, the expected revenue from the perfectly discriminating final is $\mathcal{R} = 40$.

[35] While familiar, this specific claim has so far lacked a solid foundation. For example, full rent dissipation has been considered a consequence of negative marginal payoffs at the origin, the idea being that marginal costs are positive, whereas the probability to win is supposedly constant close to the origin. That argument is circular, however, because it ignores the possibility of arbitrarily small positive bids.

[36] This CSF has been used widely in the literature. See, e.g., Clark and Riis (1998b), Hirshleifer and Osborne (2001), and Beviá and Corchón (2010).

Corollary 5. *Assume $V_1 = ... = V_n \equiv V$. Then, provided that $\rho > \frac{n}{n-1}$, it holds that $u_1^* = ... = u_n^* = 0$ and $\mathcal{R} = V$ in any symmetric equilibrium.*

The improved lower bound on ρ follows by a straightforward adaptation of the proof of Proposition 1, exploiting symmetry.[37] Needless to say, complete rent dissipation holds for homogeneous valuations in approximation in any equilibrium, symmetric or not, as a consequence of Proposition 5, provided that ρ is sufficiently large.

7.6. Dynamic settings

In dynamic contests, the principle of backwards induction implies that decisions at an earlier stage depend solely on expected payoffs in later stages. Hence, the properties of payoff and revenue equivalence stated in Proposition 1 extend naturally to any type of sequential pairwise tournament. E.g., elaborating on Groh et al. (2012), optimal seedings do not change when some noise enters the respective technologies for semifinals and finals.[38] Also the sequential multi-prize auction of Clark and Riis (1998a) can be dealt with. Under their assumptions, the *n*-player all-pay auction in each stage has a unique equilibrium, so that Proposition 2 applies if ρ is large enough.

8. Conclusion

In this paper, it has been shown that the relationship between contests with small noise and the all-pay auction is much closer than previously thought. In particular, properties that prior work had derived for *some* equilibrium of the probabilistic contest have been established, under a similar set of assumptions, for *any* equilibrium. The distinction is important because, as has been seen, even in a Tullock contest with strictly heterogeneous valuations and $R \geqslant 2$, there may be multiple, payoff-nonequivalent equilibria if the decisiveness parameter is not sufficiently high.

Although the equilibrium characterizations obtained for the probabilistic contest are overall somewhat less explicit than their counterparts for the all-pay auction, several specific applications have shown that the findings are sufficiently comprehensive to allow powerful conclusions in many settings of interest. In particular, based on the obtained findings for probabilistic contests, the analysis has established an important and general rent-dissipation result in the theory of electoral competition. We have, further, introduced with the decisiveness parameter a natural measure for the degree of noise in a probabilistic contest that might be of some independent interest. The analysis has, finally, provided a potentially useful existence result for mixed equilibria in contests with heterogeneous valuations, and has revealed new properties of the equilibrium set of the *n*-player Tullock contest.[39]

Appendix A. Proofs

This appendix is structured as follows. First, the proofs of Lemmas 1 and 2 are provided. Then, we state and prove a sequence of auxiliary results (Lemmas A.1 through A.9) that prepare the equilibrium analysis. Lemma A.1 establishes the continuity of the equilibrium payoff function at positive bid levels, and may be of independent interest. Lemmas A.2 and A.3 are key to the equilibrium analysis. Lemma A.4 captures an argument crucial for the case of two players. Lemmas A.5 through A.8 are successive steps dealing with the case of three or more players. Lemma A.9 is needed for the robustness result. Finally, we provide the proofs of Propositions 1 through 6.

Proof of Lemma 1. Suppose that, instead of maximizing u_i, each player $i = 1, ..., n$ maximizes normalized payoffs $U_i(b_i, b_{-i}) \equiv \frac{u_i(b_i, b_{-i})}{V_i} = \Psi_i(b_i, b_{-i}) - \frac{b_i}{V_i}$. Clearly, this change of units does not affect the equilibrium set. For the resulting game, however, existence of a mixed equilibrium has been shown by Bagh (2010). $\square$

Proof of Lemma 2. The case of the Tullock contest is dealt with in the text. In the case of the ratio form, $Q_i(b) = h(b_i) / \sum_{j \neq i} h(b_j)$, so that $\rho_i(b) = b_i h'(b_i)/h(b_i)$. The claim follows. Next, for the serial contest, simple calculations show that $\rho_i(b) = \alpha/(1 - \Psi_i^{SER}(b))$ if $b_i < b_j$ and $\rho_i(b) = \alpha/\Psi_i^{SER}(b)$ if $b_i \geqslant b_j$. Hence, $\rho(\Psi^{SER}) \geqslant \alpha$. But $\Psi_i^{SER}(b) \to 0$ when $b_i \to 0$ with $b_j > 0$ fixed. Therefore, $\rho(\Psi^{SER}) = \alpha$. Finally, consider the majority contest. Using Snyder (1989, Comment 4.2), one finds

$$\frac{\partial \Psi_i^{MAJ}(b; s, R)}{\partial b_i} = (2s + 1)\binom{2s}{s}(\Psi_i^{TUL}(b; R))^s (1 - \Psi_i^{TUL}(b; R))^s \frac{\partial \Psi_i^{TUL}(b; R)}{\partial b_i}. \tag{18}$$

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[37] See also the companion paper (Ewerhart, 2015, p. 69), where a similar result is obtained for the Tullock contest.

[38] Analogous conclusions may be drawn in dynamic contests considered by Rosen (1986), Agastya and McAfee (2006), Klumpp and Polborn (2006), Konrad and Kovenock (2009), Sela (2012), and Fu et al. (2015).

[39] In fact, the analysis paves the way for a complete characterization of equilibrium payoffs and expected revenue in the two-player Tullock contest with heterogeneous valuations and any $R \in (0, \infty)$, as will be reported elsewhere.

Combining this with equation (10), player i's own-bid elasticity of the odds of winning in the majority contest is seen to equal

$$\rho_i(b) = \frac{(2s+1)!}{(s!)^2} \frac{(\Psi_i^{\text{TUL}}(b;R))^{s+1}(1-\Psi_i^{\text{TUL}}(b;R))^{s+1}}{\Psi_i^{\text{MAJ}}(b;s,R)(1-\Psi_i^{\text{MAJ}}(b;s,R))} \cdot R. \tag{19}$$

Hence, $\rho_i(b) = \widehat{\rho}_i(\Psi_i^{\text{TUL}}(b;R))$, where the function $\widehat{\rho}_i : (0,1) \to \mathbb{R}_+$ is given by

$$\widehat{\rho}_i(p) = \frac{(2s+1)!}{(s!)^2} \frac{p^{s+1}(1-p)^{s+1}R}{\left(\sum_{t=s+1}^{2s+1}\binom{2s+1}{t}p^t(1-p)^{2s+1-t}\right)\left(\sum_{t=0}^{s}\binom{2s+1}{t}p^t(1-p)^{2s+1-t}\right)}. \tag{20}$$

Since $\widehat{\rho}_i(\frac{1}{2}) = \frac{(2s+1)!}{(s!)^2 4^s}R$, it remains to be checked that $\widehat{\rho}_i$ assumes its minimum at $p = \frac{1}{2}$. For this, note that differentiating $\ln\widehat{\rho}_i$ using relationship (18) yields

$$\frac{\partial \ln\widehat{\rho}_i}{\partial p} = \frac{s+1}{p} - \frac{(s+1)\binom{2s+1}{s+1}p^s(1-p)^s}{\sum_{t=s+1}^{2s+1}\binom{2s+1}{t}p^t(1-p)^{2s+1-t}} - \frac{s+1}{1-p} + \frac{(s+1)\binom{2s+1}{s}p^s(1-p)^s}{\sum_{t=0}^{s}\binom{2s+1}{t}p^t(1-p)^{2s+1-t}} \tag{21}$$

$$= \frac{s+1}{p} \cdot \frac{\sum_{t=s+2}^{2s+1}\binom{2s+1}{t}p^t(1-p)^{2s+1-t}}{\sum_{t=s+1}^{2s+1}\binom{2s+1}{t}p^t(1-p)^{2s+1-t}} - \frac{s+1}{1-p} \cdot \frac{\sum_{t=0}^{s-1}\binom{2s+1}{t}p^t(1-p)^{2s+1-t}}{\sum_{t=0}^{s}\binom{2s+1}{t}p^t(1-p)^{2s+1-t}}. \tag{22}$$

Multiplying through with the common denominator, and dividing by $(1-p)^{4s+3}$, one finds that the derivative $(\partial \ln\widehat{\rho}_i/\partial p)$ is positive if and only if

$$\left(\sum_{t=0}^{s-1}\binom{2s+1}{t+s+2}Q^t\right)\left(\sum_{t=0}^{s}\binom{2s+1}{t}Q^t\right) > \left(\sum_{t=0}^{s}\binom{2s+1}{t+s+1}Q^t\right)\left(\sum_{t=0}^{s-1}\binom{2s+1}{t}Q^t\right), \tag{23}$$

where $Q = \frac{p}{1-p}$. Multiplying out, and collecting terms, this is seen to be equivalent to the polynomial inequality $\xi(Q) \equiv \sum_{t=0}^{2s-1}\gamma_t Q^t > 0$, with negative coefficients

$$\gamma_t = \sum_{t'=0}^{t}\underbrace{\left\{\binom{2s+1}{t'+s+2} - \binom{2s+1}{t'+s+1}\right\}}_{<0}\binom{2s+1}{t-t'} < 0 \tag{24}$$

for $t \in \{0, ..., s-1\}$, and positive coefficients

$$\gamma_t = \sum_{t'=t-s}^{s-1}\binom{2s+1}{t'+s+2}\underbrace{\left\{\binom{2s+1}{t-t'} - \binom{2s+1}{t-t'-1}\right\}}_{>0} > 0 \tag{25}$$

for $t \in \{s, ..., 2s-1\}$. Hence, by Descartes' rule of signs, $\xi(Q)$ has precisely one positive root, and as $\gamma_0 < 0$, changes sign there from negative to positive. Since the right-hand side of (21) vanishes at $p = \frac{1}{2}$, this is indeed the minimum of $\widehat{\rho}_i$. $\quad\square$

Lemma A.1. *Let μ^* be a mixed equilibrium in an n-player contest. Then, for any player $i \in \{1, ..., n\}$,*

(i) *the equilibrium payoff function $b_i \mapsto E[u_i(b_i, b_{-i})|\mu_{-i}^*]$ is continuous at any $b_i > 0$, and*

(ii) *$u_i^* = E[u_i(b_i, b_{-i})|\mu_{-i}^*]$ for any $b_i \in S(\mu_i^*)\backslash\{0\}$.*

Proof. (i) Let $\{b_i^{(\nu)}\}_{\nu=1}^{\infty}$ be a sequence of bids converging to some $b_i > 0$. Then, by smoothness, the sequence $\{u_i(b_i^{(\nu)},\cdot)\}_{\nu=1}^{\infty}$ converges pointwise to $u_i(b_i,\cdot)$. Noting that $|u_i(b_i^{(\nu)},\cdot)| \leqslant \max\{V_i, b_i^{(\nu)}\}$ for any ν, and that the convergent sequence $\{b_i^{(\nu)}\}_{\nu=1}^{\infty}$ is necessarily bounded, the claim follows now directly from Lebesgue's theorem.

(ii) Clearly, $E[u_i(b_i, b_{-i})|\mu_{-i}^*] \leqslant u_i^*$ for any $b_i \in \mathbb{R}_+$. To provoke a contradiction, suppose that $E[u_i(b_i^0, b_{-i})|\mu_{-i}^*] < u_i^*$ for some $b_i^0 \in S(\mu_i^*)\backslash\{0\}$. Then, by part (i), there is an $\varepsilon > 0$ such that $E[u_i(b_i, b_{-i})|\mu_{-i}^*] < u_i^*$ for all $b_i \in \mathcal{N} \equiv (b_i^0 - \varepsilon, b_i^0 + \varepsilon)$. But $\mu_i^*(\mathcal{N}) > 0$, since $\mathcal{N}$ is an open neighborhood of $b_i^0 \in S(\mu_i^*)$. Hence, one may define a conditional probability measure through $\mu_i^-(Y_i) = \mu_i^*(Y_i \cap \mathcal{N})/\mu_i^*(\mathcal{N})$ for any measurable set $Y_i \subseteq [0, V_i]$. Moreover, $E[u_i(b_i, b_{-i})|\mu_i^-, \mu_{-i}^*] < u_i^*$. Denote the complement of $\mathcal{N}$ by $\mathcal{N}^c = [0, V_i]\backslash\mathcal{N}$. Then $\mu_i^*(\mathcal{N}^c) > 0$ because, otherwise, $\mu_i^- = \mu_i^*$, in conflict with the suboptimality of μ_i^-. Hence, one may also define $\mu_i^+(Y_i) = \mu_i^*(Y_i \cap \mathcal{N}^c)/\mu_i^*(\mathcal{N}^c)$. Since $\mu_i^* = w\mu_i^- + (1-w)\mu_i^+$, with $w \equiv \mu_i^*(\mathcal{N}) \in (0, 1)$, it follows that

$$u_i^* = w \underbrace{E[u_i(b_i, b_{-i})|\mu_i^-, \mu_{-i}^*]}_{<u_i^*} + (1 - w)E[u_i(b_i, b_{-i})|\mu_i^+, \mu_{-i}^*]. \tag{26}$$

But then, $E[u_i(b_i, b_{-i})|\mu_i^+, \mu_{-i}^*] > u_i^*$, which is the desired contradiction. $\square$

Lemma A.2. *Consider an n-player contest. Then, in any mixed equilibrium μ^*, there are players i and j with $i \neq j$ such that $\pi_i = 1$ and $\pi_j > 0$.*

Proof. Suppose that $\pi_i < 1$ for all $i = 1, ..., n$. Then player 1, say, could raise his zero bids to some small $\varepsilon > 0$, thereby increasing the probability of winning against coincident zero bids of players $2, ..., n$ from $\Psi_1(\mathbf{0}) = \frac{1}{n}$ to $\Psi_1(\varepsilon, \mathbf{0}_{-1}) = 1$. Further, if we had $\pi_j = 0$ for all $j \neq i$, then i could profitably shade his positive bids. $\square$

Lemma A.3. *Consider an n-player contest with $\rho > 2$. Then, in any mixed equilibrium μ^* with $\pi_i = 1$ for some $i \in \{1, ..., n\}$, it holds that*

(i) $\underline{b}_j = 0$ *for any $j \neq i$, and*
(ii) $u_j^* = 0$ *for any $j \neq i$.*

Proof. The proof follows closely the discussion in the body of the paper.

(i) To provoke a contradiction, suppose that $\underline{b}_j > 0$ for some $j \neq i$. There are three cases. Assume first that $\underline{b}_j \geqslant \underline{b}_i > 0$. Fix some arbitrary profile of bids $b_{-i} = (b_j, b_{-i,j})$, where $b_j \geqslant \underline{b}_j$ and $b_{-i,j} \in \mathbb{R}_+^{n-2}$. Then, from anonymity, $1 \geqslant \Psi_{i,j}(b_j, b_j, \cdot) + \Psi_{j,i}(b_j, b_j, \cdot) = 2\Psi_{i,j}(b_j, b_j, \cdot)$, so that $\Psi_{i,j}(b_j, b_j, \cdot) \leqslant \frac{1}{2}$. Hence, if i lowers his bid from b_j to $\underline{b}_i$, monotonicity implies $\Psi_{i,j}(\underline{b}_i, b_j, \cdot) \leqslant \frac{1}{2}$. Therefore, $(1 - \Psi_i(\underline{b}_i, b_{-i}))\rho_i(\underline{b}_i, b_{-i}) > 1$, or equivalently, $\Psi_i(\underline{b}_i, b_{-i})/\underline{b}_i < \partial\Psi_i(\underline{b}_i, b_{-i})/\partial b_i$. Taking expectations, and invoking Lemma A.1, one finds

$$0 \leqslant \frac{u_i^*}{\underline{b}_i} = E\left[\frac{\Psi_i(\underline{b}_i, b_{-i})}{\underline{b}_i}\bigg|\mu_{-i}^*\right]V_i - 1 < E\left[\frac{\partial\Psi_i(\underline{b}_i, b_{-i})}{\partial b_i}\bigg|\mu_{-i}^*\right]V_i - 1. \tag{27}$$

It is claimed now that the right-hand side of (27) is, in fact, weakly negative, which yields the desired contradiction. For this, consider the differential quotient $\psi_i(b_i, \cdot) \equiv \frac{\Psi_i(b_i, \cdot) - \Psi_i(\underline{b}_i, \cdot)}{b_i - \underline{b}_i}$, for $b_i > \underline{b}_i$. From monotonicity, $\psi_i(b_i, \cdot) \geqslant 0$. Moreover, since $\underline{b}_i$ is optimal, $E[\psi_i(b_i, b_{-i})|\mu_{-i}^*] \leqslant 1/V_i$. Therefore, using Fatou's Lemma,

$$E[\tfrac{\partial\Psi_i(\underline{b}_i, b_{-i})}{\partial b_i}|\mu_{-i}^*] = E[\liminf_{b_i \searrow \underline{b}_i}\psi_i(b_i, b_{-i})|\mu_{-i}^*] \leqslant \liminf_{b_i \searrow \underline{b}_i} E[\psi_i(b_i, b_{-i})|\mu_{-i}^*] \leqslant \tfrac{1}{V_i}. \tag{28}$$

Plugging the resulting inequality into (27) proves the claim and completes the first case. Assume next that $\underline{b}_i > \underline{b}_j > 0$. This case follows from the first by exchanging the roles of players i and j. Assume, finally, that $\underline{b}_i = 0$. In this case, $\pi_i = 1$ implies $\underline{b}_i^+ = 0$. Hence, there is some $b_i^+ \in S(\mu_i^*)$ with $\underline{b}_j > b_i^+ > 0$. The proof now proceeds as in the first case, with $\underline{b}_i$ replaced by b_i^+.

(ii) Consider some player $j \neq i$. Assume first that j is not always active. Then, by Assumption 2, any zero bid submitted by j loses with certainty against i. Hence, $u_j^* = 0$, as claimed. Assume next that j is always active. Since $\underline{b}_j = 0$, this implies $\underline{b}_j^+ = 0$. Hence, there is a sequence $\{b_j^{(\nu)}\}_{\nu=1}^\infty$ in $S(\mu_j^*)\backslash\{0\}$ such that $\lim_{\nu\to\infty} b_j^{(\nu)} = 0$. By Lemma A.1, $u_j^* = E[u_j(b_j^{(\nu)}, b_{-j})|\mu_{-j}^*]$, for any ν. Hence, by Lebesgue's theorem, $u_j^* = E[\lim_{\nu\to\infty} u_j(b_j^{(\nu)}, b_{-j})|\mu_{-j}^*]$. If now $b_{-j} \neq \mathbf{0}_{-j}$, then Assumptions 4 and 2 imply that $\lim_{\nu\to\infty} u_j(b_j^{(\nu)}, b_{-j}) = u_j(0, b_{-j}) = 0$. If, however, $b_{-j} = \mathbf{0}_{-j}$, then by Assumption 2, $\lim_{\nu\to\infty} u_j(b_j^{(\nu)}, \mathbf{0}_{-j}) = \lim_{\nu\to\infty}(V_j - b_j^{(\nu)}) = V_j$. It follows that $u_j^* = V_j \cdot \prod_{k\neq j}(1 - \pi_k)$. But $\pi_i = 1$ and $i \neq j$, so that $u_j^* = 0$, as claimed. $\square$

Lemma A.4. *Consider a mixed equilibrium $\mu^* = (\mu_1^*, \mu_2^*)$ in a two-player contest such that $u_1^* = u_2^* = 0$. Then, $V_1 = V_2$ and $p_1^* = p_2^* = \frac{1}{2}$.*

Proof. Since $u_2^* = 0$, a deviation by player 2 to player 1's strategy μ_1^* returns a weakly negative expected payoff, i.e., $\frac{V_2}{2} - b_1^* \leqslant 0$. Combining this with $u_1^* = p_1^*V_1 - b_1^* = 0$ delivers $p_1^*V_1 \geqslant \frac{V_2}{2}$. By symmetry, $p_2^*V_2 \geqslant \frac{V_1}{2}$. Summing up, one obtains

$$p_1^*V_1 + p_2^*V_2 \geqslant \frac{V_1 + V_2}{2}, \tag{29}$$

with $p_1^* + p_2^* = 1$. If $V_1 > V_2$, then (29) implies $p_2^* \leqslant \frac{1}{2}$, so that, using $p_2^*V_2 \geqslant \frac{V_1}{2}$, one arrives at $V_1 \leqslant V_2$, a contradiction. The case $V_1 < V_2$ is similar. Hence, $V_1 = V_2$, and (29) is an equality. But then, $p_1^*V_1 \geqslant \frac{V_2}{2}$ and $p_2^*V_2 \geqslant \frac{V_1}{2}$ must be equalities as well. Thus, also $p_1^* = p_2^* = \frac{1}{2}$. $\square$

Lemma A.5. $\Psi_{i,j}(b_i, 0, b_{-i,j}) \leqslant 2\Psi_{i,j}(b_i, b_i, b_{-i,j})$, for any $(b_i, b_{-i,j}) \in \mathbb{R}_+^{n-1}$.

Proof. Since winning probabilities across players sum up to one,

$$\Psi_{i,j}(b_i, b_j, b_{-i,j}) + \Psi_{j,i}(b_j, b_i, b_{-j,i}) + \sum_{k \neq i,j} \Psi_{k,j}(b_k, b_j, b_{-k,j}) = 1, \tag{30}$$

for any $b \in \mathbb{R}_+^n$. Evaluating at $b_j = b_i$, anonymity and monotonicity imply

$$2\Psi_{i,j}(b_i, b_i, b_{-i,j}) = 1 - \sum_{k \neq i,j} \Psi_{k,j}(b_k, b_i, b_{-k,j}) \tag{31}$$

$$\geqslant 1 - \sum_{k \neq i,j} \Psi_{k,j}(b_k, 0, b_{-k,j}) \tag{32}$$

$$\geqslant 1 - \Psi_{j,i}(0, b_i, b_{-j,i}) - \sum_{k \neq i,j} \Psi_{k,j}(b_k, 0, b_{-k,j}). \tag{33}$$

Using now (30) again with $b_j = 0$, the lemma follows. □

Lemma A.6. Fix $\kappa > 1$. Then, provided that ρ is sufficiently large, $\Psi_{i,j}(\kappa b_j, b_j, b_{-i,j}) > \Psi_{i,j}(b_j, 0, b_{-i,j})/\kappa$ for any $b_j > 0$ and any $b_{-i,j} \in \mathbb{R}_+^{n-2}$.

Proof. Suppose that $\Psi_{i,j}(\kappa b_j, b_j, b_{-i,j}) \leqslant \Psi_{i,j}(b_j, 0, b_{-i,j})/\kappa$ for some $b_j > 0$ and some $b_{-i,j} \in \mathbb{R}_+^{n-2}$. Then, obviously, $\Psi_{i,j}(\kappa b_j, b_j, b_{-i,j}) \leqslant 1/\kappa$, and hence, by monotonicity, $\Psi_{i,j}(\tilde{\kappa} b_j, b_j, b_{-i,j}) \leqslant 1/\tilde{\kappa}$ for any $\tilde{\kappa} \in [1, \kappa]$. Writing $b_{-i} = (b_j, b_{-i,j})$, this implies

$$\frac{\partial \ln \Psi_i(\tilde{\kappa} b_j, b_{-i})}{\partial \ln \tilde{\kappa}} = (1 - \Psi_i(\tilde{\kappa} b_j, b_{-i}))\rho_i(\tilde{\kappa} b_j, b_{-i}) \geqslant (1 - \frac{1}{\tilde{\kappa}})\rho \tag{34}$$

for any $\tilde{\kappa} \in [1, \kappa]$. Consequently, using the fundamental theorem of calculus,

$$\ln\left(\frac{\Psi_i(\kappa b_j, b_{-i})}{\Psi_i(b_j, b_{-i})}\right) = \int_1^\kappa \frac{\partial \ln \Psi_i(\tilde{\kappa} b_j, b_{-i})}{\partial \ln \tilde{\kappa}} \frac{d\tilde{\kappa}}{\tilde{\kappa}} \geqslant \underbrace{(\ln \kappa + \frac{1}{\kappa} - 1)}_{>0} \rho. \tag{35}$$

Since, for ρ large, the right-hand side of (35) exceeds all bounds, $\Psi_{i,j}(\kappa b_j, b_j, b_{-i,j}) > \frac{2}{\kappa}\Psi_{i,j}(b_j, b_j, b_{-i,j})$. Lemma A.5 with $b_i = b_j$ leads then to the desired contradiction. □

Lemma A.7. Consider an n-player contest, and let $\kappa > 1$. Then, for any $i \neq j$, and for ρ sufficiently large, $E[\Psi_i(\kappa \bar{b}_j, b_{-i})|\mu^*_{-i}] > E[\Psi_j(\bar{b}_j, b_{-j})|\mu^*_{-j}]/\kappa$ in any mixed equilibrium μ^* with $\bar{b}_j > 0$.

Proof. For ρ large enough, monotonicity and Lemma A.6 imply $\Psi_{i,j}(\kappa \bar{b}_j, b_j, \cdot) \geqslant \Psi_{i,j}(\kappa \bar{b}_j, \bar{b}_j, \cdot) > \Psi_{i,j}(\bar{b}_j, 0, \cdot)/\kappa$ for any $b_j \in S(\mu^*_j)$. Moreover, by anonymity and monotonicity, $\Psi_{i,j}(\bar{b}_j, 0, \cdot) = \Psi_{j,i}(\bar{b}_j, 0, \cdot) \geqslant \Psi_{j,i}(\bar{b}_j, b_i, \cdot)$ for any $b_i \in \mathbb{R}_+$. Hence, $\Psi_{i,j}(\kappa \bar{b}_j, b_j, \cdot) > \Psi_{j,i}(\bar{b}_j, b_i, \cdot)/\kappa$. Taking expectations w.r.t. μ^*, the claim follows. □

Lemma A.8. Fix $V_i > V_j$, for some $i \neq j$. Then, in any n-player contest with ρ sufficiently large, $\pi_j > 0$ implies $u_i^* > 0$.

Proof. Since $\pi_j > 0$, clearly $\bar{b}_j > 0$. Hence, Lemma A.1 implies $E\left[\Psi_j(\bar{b}_j, b_{-j})\big|\mu^*_{-j}\right] V_j - \bar{b}_j = u_j^* \geqslant 0$. Multiplying through with $\kappa \equiv \sqrt{V_i/V_j} > 1$ delivers $E\left[\frac{\Psi_j(\bar{b}_j, b_{-j})}{\kappa}\big|\mu^*_{-j}\right] V_i - \kappa \bar{b}_j \geqslant 0$. From Lemma A.7, $E\left[\Psi_i(\kappa \bar{b}_j, b_{-i})\big|\mu^*_{-i}\right] V_i - \kappa \bar{b}_j > 0$, i.e., player i's expected payoff from choosing the bid level $b_i = \kappa \bar{b}_j$ against μ^*_{-i} is positive. Since the equilibrium strategy μ^*_i must perform at least as well as b_i, it follows that $u_i^* > 0$. □

Lemma A.9. Let $b_i > 0$ and $\varepsilon > 0$. Then, for ρ sufficiently large, it holds that $\Psi_i(b_i(1+\varepsilon), b_{-i}) \geqslant 1 - \varepsilon$ for any $b_{-i} \in \mathbb{R}_+^{n-1}$ satisfying $b_j \leqslant b_i$ for all $j \neq i$.

Proof. If $\Psi_i(b_i, b_{-i}) = 1$, then the claim follows directly from monotonicity. Assume, therefore, that $\Psi_i(b_i, b_{-i}) < 1$. Then, monotonicity and anonymity imply $\Psi_i(b_i, b_{-i}) \geqslant \Psi_i(b_i, ..., b_i) = \frac{1}{n}$. Hence, player i's odds of winning satisfy $Q_i(b_i, b_{-i}) = \frac{\Psi_i(b_i, b_{-i})}{1 - \Psi_i(b_i, b_{-i})} \geqslant \frac{1}{n-1}$. If i raises his bid to $b_i^\# = (1+\varepsilon)b_i$, then by the fundamental theorem of calculus,

$$\ln\left(\frac{Q_i(b_i^\#, b_{-i})}{Q_i(b_i, b_{-i})}\right) = \int_{b_i}^{b_i^\#} \rho_i(\widetilde{b}_i, b_{-i}) \frac{d\widetilde{b}_i}{\widetilde{b}_i} \geqslant \rho \ln(1+\varepsilon). \tag{36}$$

Applying the exponential function delivers

$$Q_i(b_i^\#, b_{-i}) \geqslant (1+\varepsilon)^\rho \cdot Q_i(b_i, b_{-i}) \geqslant (1+\varepsilon)^\rho \cdot \frac{1}{n-1}. \tag{37}$$

But the right-hand side of inequality (37) grows above all bounds as $\rho \to \infty$, and so do player i's odds of winning $Q_i(b_i^\#, b_{-i})$, regardless of b_{-i}. Therefore, for ρ large enough, $\Psi_i(b_i^\#, b_{-i}) = \frac{Q_i(b_i^\#, b_{-i})}{1+Q_i(b_i^\#, b_{-i})} \geqslant 1-\varepsilon$, as claimed. $\square$

Proof of Proposition 1. Let $\mu^* = (\mu_1^*, \mu_2^*)$ be any mixed equilibrium in the two-player contest. By Lemma A.2, there are indices $i, j \in \{1, 2\}$ with $i \neq j$ such that $\pi_i = 1$ and $\pi_j \in (0, 1]$. We now reverse the trick of Alcalde and Dahm (2010). For this, consider the *modified contest* in which the always active player's valuation V_i is weakly scaled down to $V_i' = \pi_j V_i \in (0, V_i]$. Define a mixed strategy $\widehat{\mu}_j$ in the modified contest via $\widehat{\mu}_j(Y_j) = \mu_j^*(Y_j \backslash \{0\})/\pi_j$, for any measurable set $Y_j \subseteq [0, V_j]$. Intuitively, strategy $\widehat{\mu}_j$ samples only from the active part of μ_j^*. It is straightforward to check that $(\mu_i^*, \widehat{\mu}_j)$ is a mixed equilibrium in the modified contest. In that equilibrium, both players are always active. As a consequence, by Lemma A.3, both players use arbitrarily small positive bids and have a zero rent in the modified contest. Lemma A.4 implies now that $V_j = V_i'$, where $V_i' = \pi_j V_i$ by construction. Hence, $\pi_j = V_j/V_i$. Yet π_j is a probability, so that $V_j \leqslant V_i$. Thus, $i = 1$ and $j = 2$ (necessarily so if $V_1 > V_2$, and without loss of generality if $V_1 = V_2$). It follows that player 2 participates with probability $\pi_2 = V_2/V_1$ in the original contest. Moreover, in the modified contest, by Lemma A.4, each player wins with probability $1/2$ and, consequently, bids on average $V_2/2$. In sum, player 1 realizes in the original contest a rent of $u_1^* = \pi_2 \frac{V_1}{2} + (1 - \pi_2)V_1 - \frac{V_2}{2} = V_1 - V_2$. The remaining claims are now immediate. $\square$

Proof of Proposition 2. Suppose there is a mixed equilibrium μ^* in which some player $j \in \{3, ..., n\}$ is active. Then, since $V_1 \geqslant V_2 > V_3 \geqslant V_j$, Lemma A.8 implies $u_1^* > 0$ and $u_2^* > 0$, provided that ρ is large enough. From Lemmas A.2 and A.3, however, there can be at most one player with a positive rent in this case, which yields a contradiction. The assertion follows now from Proposition 1. $\square$

Proof of Proposition 3. By Lemmas A.2 and A.3, some player i is always active, and $u_j^* = 0$ for all $j \neq i$. Hence, provided that ρ is large enough, Lemma A.8 implies $V_1 = V_i$, or equivalently, $i \leqslant m$. Since, by assumption, at least two (even three) players have a valuation of V_1, a similar argument shows that $\pi_j = 0$ for any $j = m+1, ..., n$. As for the revenue, note that $u_j^* = p_j^* V_1 - b_j^*$ for all $j = 1, ..., m$. Summing up yields $u_i^* = V_1 - \mathcal{R}$. Finally, if the equilibrium is "symmetric," then necessarily $u_1^* = ... = u_m^*$. Since $m > 1$, this is only feasible if $u_i^* = 0$. $\square$

Proof of Proposition 4. By Lemma A.2, there are players i and j with $j \neq i$ such that $\pi_i = 1$ and $\pi_j > 0$. Since either $i > 1$ or $j > 1$, Lemma A.8 implies $u_1^* > 0$. By Lemma A.3, this is only possible if $i = 1$ and $u_2^* = ... = u_n^* = 0$. Next, for any $k \geqslant m+1$, the inequality $V_2 > V_k$ and Lemma A.8 imply that $\pi_k = 0$. As for the expected revenue, note that $b_1^* = p_1^* V_1 - u_1^*$, and that $b_k^* = p_k^* V_2$ for $k = 2, ..., m$. Summing up yields $\mathcal{R} = V_2 + p_1^*(V_1 - V_2) - u_1^*$, as claimed. It remains to prove $\mathcal{R} < V_2$. For this, fix some small $\delta > 0$. By Proposition 5 below,[40] we have $u_1^* > (1 - \delta)(V_1 - V_2)$ if ρ is large enough. It therefore suffices to show that, for ρ large, $p_1^* \leqslant 1 - \delta$ in any mixed equilibrium. Suppose that, instead, $p_1^* > 1 - \delta$ in some equilibrium μ^*. Then, obviously, $p_j^* < \delta$ for all $j = 2, ..., m$. But, $u_j^* = p_j^* V_2 - b_j^* = 0$, so that $b_j^* < \delta V_2$. Hence, $\text{pr}(b_j \geqslant \frac{V_2}{2}) < 2\delta$. It follows that $\text{pr}(b_j < \frac{V_2}{2}$ for $j = 2, ..., m) > (1 - 2\delta)^{m-1}$. Fix now some $\varepsilon > 0$. Then, for ρ large, Lemma A.9 implies that bidding $\frac{(1+\varepsilon)V_2}{2}$ yields player 1 an expected payoff strictly above $(1 - \varepsilon)(1 - 2\delta)^{m-1} V_1 - \frac{(1+\varepsilon)V_2}{2}$. However, for δ and ε small enough, this is strictly larger than $(1 + \delta)(V_1 - V_2)$, a contradiction. This completes the proof of the proposition. $\square$

Proof of Proposition 5. Fix valuations $V_1 \geqslant V_2 \geqslant ... \geqslant V_n > 0$ and $\delta > 0$. It is shown first that, for ρ sufficiently large, $u_1^* \geqslant V_1 - V_2 - \delta$ in any equilibrium μ^*. To see this, let $b_1^\# = (1+\varepsilon)V_2$, for some $\varepsilon > 0$. Then, by Lemma A.9, for ρ sufficiently large, $\Psi_1(b_1^\#, b_{-1}) \geqslant 1 - \varepsilon$ holds for any $b_{-1} \in [0, V_2]^{n-1}$. Hence,

$$u_1^* \geqslant E[u_1(b_1^\#, b_{-1}) | \mu_{-1}^*] \geqslant (1-\varepsilon)V_1 - (1+\varepsilon)V_2 = V_1 - V_2 - \varepsilon(V_1 + V_2). \tag{38}$$

Choosing $\varepsilon = \delta/(V_1 + V_2)$, the claim follows. Next, it is shown that, for ρ sufficiently large, $u_1^* \leqslant V_1 - V_2 + \delta$ in any mixed equilibrium. To provoke a contradiction, suppose that $u_1^* > V_1 - V_2 + \delta$ in some equilibrium μ^*. Then, $u_1^* > 0$ and, hence, via Lemma A.2, $\bar{b}_1 > 0$. Therefore, by Lemma A.1, $u_1^* = E[\Psi_1(\bar{b}_1, b_{-1}) | \mu_{-1}^*]V_1 - \bar{b}_1$. It follows that $E[\Psi_1(\bar{b}_1, b_{-1}) | \mu_{-1}^*]V_2 - \bar{b}_1 \geqslant$

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[40] As explained in the body of the paper, this use of a later result is for expositional purposes only and does not constitute any circularity.

$u_1^* - (V_1 - V_2) > \delta$. Choose $\varepsilon > 0$ such that $(\varepsilon^2 + 2\varepsilon)V_1 < \delta$. Then $E[\Psi_1(\bar{b}_1, b_{-1})|\mu_{-1}^*]V_2 - (1 + \varepsilon)^2 \bar{b}_1 > 0$. For ρ large enough, Lemma A.7 implies now $E[\Psi_2(b_2^\#, b_{-2})|\mu_{-2}^*]V_2 - b_2^\# > 0$, where $b_2^\# = (1 + \varepsilon)\bar{b}_1$. Thus, $u_2^* > 0$, which is impossible in view of Lemma A.3. The claim follows. It remains to be shown that $u_j^* < \delta$ for any $j \neq 1$. There are two cases. If $V_1 = V_j$, then, after exchanging the roles of players 1 and j, the claim follows from earlier arguments. If, however, $V_1 > V_j$, then $u_j^* = 0$ from Propositions 2, 3 and 4. This completes the proof. □

Proof of Proposition 6. Fix valuations $V_1 > V_2 > ... > V_n$, and some $R \geqslant 2$. Assume for the moment that player 1 remains passive. Then, as shown by Alcalde and Dahm (2010), there exists an all-pay auction equilibrium $(\mu_2^*, ..., \mu_n^*)$ in the $(n-1)$-player contest between players 2 through n. In that equilibrium, only players 2 and 3 are active. Suppose now that player 1 considers entering the active contest with a bid $b_1 > 0$. If player 1 were to face only μ_2^* (i.e., ignoring player 3), then player 1 would win with probability $p_{12} = E\left[\dfrac{b_1^R}{b_1^R + b_2^R} \,\middle|\, \mu_2^*\right] > 0$, and have an expected payoff of $u_{12}^* = p_{12}V_1 - b_1$. We claim that $u_{12}^* \leqslant p_{12}(V_1 - V_3)$. Suppose not. Then, in the smaller contest, player 3's expected payoff from a bid equal to b_1 (instead of μ_3^*) would amount to $p_{12}V_3 - b_1 = p_{12}V_1 - b_1 - p_{12}(V_1 - V_3) > 0$, contradicting the fact that $(\mu_2^*, ..., \mu_n^*)$ is an all-pay auction equilibrium. Thus, indeed, $u_{12}^* \leqslant p_{12}(V_1 - V_3)$. Note next that player 1's probability of winning, once μ_3^* is taken into account, is lowered by

$$E\left[\frac{b_1^R}{b_1^R + b_2^R} - \frac{b_1^R}{b_1^R + b_2^R + b_3^R} \,\middle|\, \mu_2^*, \mu_3^*\right] = E\left[\frac{b_1^R}{b_1^R + b_2^R}\left(1 - \frac{b_1^R + b_2^R}{b_1^R + b_2^R + b_3^R}\right) \,\middle|\, \mu_2^*, \mu_3^*\right] \tag{39}$$

$$= E\left[\frac{b_1^R}{b_1^R + b_2^R} E\left[\frac{b_3^R}{b_1^R + b_2^R + b_3^R} \,\middle|\, \mu_3^*\right] \,\middle|\, \mu_2^*\right] \tag{40}$$

$$\geqslant E\left[\frac{b_1^R}{b_1^R + b_2^R} \,\middle|\, \mu_2^*\right] \cdot \frac{E[\,b_3^R \mid \mu_3^*\,]}{3V_1^R} \tag{41}$$

$$\geqslant p_{12} \cdot \frac{(E[\,b_3 \mid \mu_3^*\,])^R}{3V_1^R} \tag{42}$$

$$= p_{12} \cdot \frac{1}{3V_1^R} \cdot \left(\frac{V_3^2}{2V_2}\right)^R \tag{43}$$

$$> p_{12} \cdot \frac{1}{2^{R+2}} \cdot \left(\frac{V_3}{V_1}\right)^{2R}, \tag{44}$$

where Jensen's inequality has been used. Therefore, player 1's expected payoff from bidding b_1 in the n-player contest against μ_2^* and μ_3^* satisfies

$$E\left[\frac{b_1^R V_1}{b_1^R + b_2^R + b_3^R} - b_1 \,\middle|\, \mu_2^*, \mu_3^*\right] < p_{12} \cdot \left\{V_1 - V_3 - \frac{V_1}{2^{R+2}} \cdot \left(\frac{V_3}{V_1}\right)^{2R}\right\}. \tag{45}$$

If V_1 and V_3 are sufficiently close to each other, so that $V_1 - V_3 < V_1/2^{R+3}$ and $(V_3/V_1)^{2R} > 1/2$, then the right-hand side of (45) is easily seen to be negative. Thus, with μ_1^* giving all weight to the zero bid, $(\mu_1^*, \mu_2^*, ..., \mu_n^*)$ becomes an equilibrium in the n-player contest. Moreover, $u_2^* = V_2 - V_3 > 0$, which is impossible in an all-pay auction equilibrium. Finally, it is noted that $\mathcal{R} = \frac{V_3(V_2 + V_3)}{2V_2}$ differs from $\mathcal{R}^{\text{APA}} = \frac{V_2(V_1 + V_2)}{2V_1}$ unless $V_1 V_3(V_2 + V_3) = V_2^2(V_1 + V_2)$. Hence, $\mathcal{R} \neq \mathcal{R}^{\text{APA}}$ on an nonempty open set of valuation vectors. This completes the proof. □

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